

# Height growth on semisimple groups

Anton Deitmar & Rupert McCallum

**Abstract:** A condition is given, under which a general lattice point counting function is asymptotic to the corresponding ball volume growth function. This is then used to give height asymptotics in the style of the Batyrev-Manin Conjecture for certain intrinsically defined heights on semisimple groups.

## Contents

1	Lattice points and ball volume	3
2	Small overlaps	10
3	The adelic metric space	15
4	Lattice points of $\mathbb{Q}$ -groups	18
5	The group $\mathrm{PGL}(d)$	21

## Introduction

Let  $V$  be a variety over  $\mathbb{Q}$ . A line bundle over  $V$  gives rise to a height function  $h$  which can be viewed as a means of measuring how “many” rational points the variety possesses. The Batyrev-Manin conjecture [BM90], states an asymptotic formula for the number of points of height  $\leq x$ . It has been shown to hold for flag varieties in [FMT89] and for the wonderful compactification of a semisimple linear algebraic group in [GMO08]. A linear algebraic group  $\mathbb{G}$  is an affine variety and there is no “natural” height function, so one has to construct a height via extra data, such as a projective embedding.

In her doctoral thesis under the supervision of Victor Batyrev [Men15], Susanne Mennecke proposed a more intrinsic definition of a height function and asked for an analogue of the Batyrev-Manin conjecture. This height function has the advantage of being built from data given by the group  $\mathbb{G}$  itself, like its symmetric space and the Bruhat-Tits buildings of the groups of  $p$ -adic points. The height function  $h$  is defined as the product of local heights  $h_v$ , which in turn are defined for any place  $v$  by

$$\log_v h_v(x) = \text{dist}_v(xK_v, K_v),$$

where  $\log_v$  is the natural logarithm if  $v$  is archimedean and equals the logarithm to the basis  $q_v$  otherwise, where  $q_v$  is the cardinality of the residue class field. The point  $K_v$  is the base point in the symmetric space in the archimedean case and the base point of the Bruhat-Tits building otherwise. The distance function is the combinatorial distance in the nonarchimedean case and the Riemannian distance if  $v$  is archimedean. Then the global height  $h$  on the adelic points equals

$$h(x) = \prod_v h_v(x_v).$$

Let  $\pi = \pi_{\mathbb{G}}$  denote the height counting function,

$$\pi(x) = \#\left\{\gamma \in \mathbb{G}(\mathbb{Q}) : h(\gamma) \leq x\right\}.$$

Mennecke showed that for the group  $\mathbb{G} = \text{SL}_2$  one obtains

$$\pi(x) \sim cx^{3/2}$$

for some  $c > 0$ . She conjectured a similar asymptotic for  $\mathbb{G} = \text{SL}_d$ ,  $d \geq 3$ . In the current paper we approach this problem via a general result on lattice-point problems on metric spaces, relating the lattice point counting to metric

ball growth asymptotics. In order to do computations we modify Mennecke's definition in two ways. First we replace the Riemannian distance at the infinite place by a metric which has canonical Weyl-group invariant polygons as metric balls, a change that seems natural since it matches up with the combinatorial distance at the finite places. Next we scale the metric at infinity by a free parameter which determines the dominance of either the volume growth at the finite, or the infinite places. Our result for a general group reads

$$\pi(x) \sim \frac{C}{\text{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} (B \log x)^{r_{\mathbb{R}}-1} x^B,$$

where  $B$  is the scaling constant,  $r_{\mathbb{R}}$  is the real rank of the group and  $C$  is an explicit constant. This result holds if the scaling constant  $B$  is large. For small values of  $B$  a general result of this type is harder to come by, since the volume growth  $L$ -function is not well behaved in this case. In the case of the rank one groups  $\text{PGL}_2$  or  $\text{SL}_2$  we can derive an explicit result confirming Mennecke's computations as follows: We find that for the rank one group  $\mathbb{G} = \text{PGL}_2$  or  $\mathbb{G} = \text{SL}_2$  the volume  $b^{\text{fin}}(T)$  of a ball of radius  $T > 0$  satisfies

$$b^{\text{fin}}(T) \sim ce^{dT},$$

where, in the case  $\mathbb{G} = \text{SL}_2$  we have  $c = \frac{15}{2\pi^2}$  and  $d = \frac{3}{2}$ , whereas for  $\mathbb{G} = \text{PGL}_2$  we get  $c = \frac{15}{\pi^2}$  and  $d = 2$ . If we then scale the metric at infinity with a scaling parameter  $0 < B < d$ , then we get, as  $x \rightarrow \infty$ ,

$$\pi(x) \sim cd \int_0^\infty e^{-dt} b^\infty(t) dt \frac{1}{\text{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} x^d.$$

As for the wonderful compactification the local heights are at almost all places given by formula similar to ours (see Lemma 6.4 in [STBT07]), these results are not too far from what has been done in the papers [GMO08, STBT07, TBT13].

## 1 Lattice points and ball volume

Recall the notion of a *matrix coefficient* of a unitary representation  $(\pi, V_\pi)$  of a locally compact group  $G$  on some Hilbert space  $V_\pi$ : this is any function of the form  $\psi_{v,w} : G \rightarrow \mathbb{C}$ ,  $x \mapsto \langle \pi(x)v, w \rangle$  for some  $v, w \in V_\pi$ .

For a compact subgroup  $K \subset G$  we write  $V_\pi^K$  for the set of  $K$ -invariant vectors in  $V_\pi$ . We say that a matrix coefficient  $\psi$  is  $K$ -invariant, if  $\psi(kx) =$

$\psi(xk) = \psi(x)$  holds for every  $x \in G$  and all  $k \in K$ . If this is the case, then  $\psi = \psi_{v,w}$  for some  $v, w \in V_\pi^K$ . To see this, assume that  $\psi = \psi_{\tilde{v}, \tilde{w}}$  is  $K$ -invariant, then with the normalised Haar measure on  $K$  we get  $\psi = \psi_{v,w}$  with  $v = \int_K \pi(k) \tilde{v} dk$  and  $w = \int_K \pi(k) \tilde{w} dk$ .

**Definition 1.1.** Let  $G$  be a locally compact group with a fixed Haar measure  $dx$ . Let  $\Gamma \subset G$  be a lattice, i.e.,  $\Gamma$  is a discrete subgroup such that on  $\Gamma \backslash G$  there exists a  $G$ -invariant Radon measure of finite volume. Let  $L_0^2(\Gamma \backslash G)$  denote the orthogonal space in  $L^2(\Gamma \backslash G)$  of the one-dimensional space of constant functions, i.e.,

$$L_0^2(\Gamma \backslash G) = \left\{ f \in L^2(\Gamma \backslash G) : \int_{\Gamma \backslash G} f(x) dx = 0 \right\}.$$

Let  $K \subset G$  be a compact subgroup. We say that  $G$  acts *K-mixingly* on  $\Gamma \backslash G$ , if every  $K$ -invariant matrix coefficient in the Hilbert space  $L_0^2(\Gamma \backslash G)$  vanishes at infinity.

**Definition 1.2.** Let  $b : [0, \infty) \rightarrow [0, \infty)$  be monotonically increasing with  $\lim_{T \rightarrow \infty} b(T) = \infty$ . We say that  $b$  is *regular*, if

$$\lim_{\varepsilon \searrow 0} \liminf_{T \rightarrow \infty} \frac{b(T - \varepsilon)}{b(T)} = \lim_{\varepsilon \searrow 0} \limsup_{T \rightarrow \infty} \frac{b(T + \varepsilon)}{b(T)}.$$

If this happens, then both limits are equal to 1. Note that regularity only depends on the asymptotic class, i.e., if  $b(x) \sim b_1(x)$  as  $x \rightarrow \infty$ , then  $b_1$  is regular if and only if  $b$  is regular.

**Example 1.3.** For any given  $a, b \geq 0$ , not both zero, the function  $b(x) = x^a e^{bx}$  is regular. The function  $b(x) = e^{x^2}$  is not. Also, the function  $b(x) = e^{[x]}$  is not regular, where  $[x]$  is the largest integer  $\leq x$ .

**Theorem 1.4.** Let  $G$  be a locally compact group with a lattice  $\Gamma \subset G$ . Let  $K \subset G$  be a compact group and assume that on  $X = G/K$  there is given a  $G$ -invariant proper metric  $d$ . Let  $b(T)$  denote the Haar measure of the closed ball  $\overline{B}_T$  of radius  $T > 0$  around  $eK$  in  $X$ . Let

$$N(T) = \#\left\{ \gamma \in \Gamma : d(\gamma K, eK) \leq T \right\}.$$

Assume that

(a)  $G$  acts *K-mixingly* on  $\Gamma \backslash G$  and

(b)  $b(T)$  is regular.

Then we have, as  $T \rightarrow \infty$ ,

$$N(T) \sim \frac{1}{\text{vol}(\Gamma \backslash G)} b(T).$$

It may happen that the function  $b$  is not regular per se, but there exists an unbounded subset  $\mathcal{S} \subset (0, \infty)$ , such that  $b$  is regular on  $\mathcal{S}$ , i.e., one has

$$\lim_{\varepsilon \searrow 0} \liminf_{\substack{T \rightarrow \infty \\ T \in \mathcal{S}}} \frac{b(T - \varepsilon)}{b(T)} = \lim_{\varepsilon \searrow 0} \limsup_{\substack{T \rightarrow \infty \\ T \in \mathcal{S}}} \frac{b(T + \varepsilon)}{b(T)}.$$

An example is the function  $b(x) = e^{[x]}$ . In that case, the asymptotic assertion remains true when restricted to  $\mathcal{S}$ , i.e., one then has  $\lim_{\substack{T \rightarrow \infty \\ T \in \mathcal{S}}} \frac{N(T)}{b(T)} \text{vol}(\Gamma \backslash G) = 1$ , which we denote as

$$N(T) \sim_{\mathcal{S}} \frac{1}{\text{vol}(\Gamma \backslash G)} b(T).$$

*Proof.* Fix some smooth, monotonically decreasing function  $\chi : \mathbb{R} \rightarrow \mathbb{R}$  such that

- $0 \leq \chi \leq 1$ ,
- $\chi(x) = 1$ , if  $x < -1$ ,
- $\chi(x) = 0$ , if  $x > 1$ .

For  $T > 1$  let  $\chi_T(x) = \chi(e^T(x - T))$ . Then we in particular have

$$\chi_T(x) = \begin{cases} 1 & x \leq T - e^{-T}, \\ 0 & x \geq T + e^{-T}. \end{cases}$$

**Lemma 1.5.** *For every  $\varepsilon > 0$  there is  $T_0$  such that for  $T \geq T_0$  we have*

$$\chi_{T-\varepsilon}(x) \leq \chi_T(x + \varepsilon) \quad \text{and} \quad \chi_T(x - \varepsilon) \leq \chi_{T+\varepsilon}(x).$$

*Proof.* The second claim follows from the first, so we only show the first. This claim is equivalent to

$$\chi(e^{T-\varepsilon}(x - T + \varepsilon)) \leq \chi(e^T(x - T - \varepsilon)).$$

If  $e^{T-\varepsilon}(x - T + \varepsilon) \geq 1$ , then the left hand side is zero and the claim follows. Otherwise, we have  $x \leq e^{\varepsilon-T} + T - \varepsilon$  and so

$$\begin{aligned} (e^\varepsilon - 1)x - e^\varepsilon \varepsilon &\leq (e^\varepsilon - 1)(e^{\varepsilon-T} + T - \varepsilon) - e^\varepsilon \varepsilon \\ &= (e^\varepsilon - 1)T + \underbrace{((e^\varepsilon - 1)(e^{\varepsilon-T} - \varepsilon) - e^\varepsilon \varepsilon)}_{<0 \text{ for } T \ll 0}. \end{aligned}$$

The second summand has a limit  $< 0$  as  $T \rightarrow \infty$ , therefore there exists  $T_0$  such that this summand is  $< \varepsilon$  for  $T \geq T_0$ . Therefore, for  $T \geq T_0$  we have

$$(e^\varepsilon - 1)x - e^\varepsilon \varepsilon \leq (e^\varepsilon - 1)T + \varepsilon,$$

or

$$e^T(x - T - \varepsilon) \leq e^{T-\varepsilon}(x - T + \varepsilon)$$

As  $\chi$  is monotonically decreasing, the claim follows.  $\square$

Define a function  $f_T : G \rightarrow \mathbb{C}$  by  $f_T(x) = \chi_T(d(xK, K))$  and let  $k_T(x, y) = \sum_{\gamma \in \Gamma} f_T(x^{-1}\gamma y)$ . Then for any  $\phi \in L^2(\Gamma \backslash G)$  we use the quotient integral formula [DE14, Chapter 1] to compute

$$\begin{aligned} R(f_T)\phi(x) &= \int_G f_T(y)\phi(xy) dy \\ &= \int_G f_T(x^{-1}y)\phi(y) dy \\ &= \int_{\Gamma \backslash G} \sum_{\gamma \in \Gamma} f_T(x^{-1}\gamma y)\phi(y) dy \\ &= \int_{\Gamma \backslash G} k_T(x, y)\phi(y) dy. \end{aligned}$$

So  $k_T \in C(\Gamma \backslash G \times \Gamma \backslash G)$  is the integral kernel of the operator  $R(f_T)$ .

Let  $\lambda_T$  denote the eigenvalue of  $R(f_T)$  on the space of constant functions. The latter has the function  $\phi_{\text{const}} \equiv \frac{1}{\sqrt{\text{vol}(\Gamma \backslash G)}}$  for an orthonormal basis. We have

$$\begin{aligned} \lambda_T &= \sqrt{\text{vol}(\Gamma \backslash G)} \lambda_T \phi_{\text{const}}(1) \\ &= \sqrt{\text{vol}(\Gamma \backslash G)} R(f_T) \phi_{\text{const}}(1) \\ &= \sqrt{\text{vol}(\Gamma \backslash G)} \int_G f_T(x) \phi_{\text{const}}(x) dx \\ &= \int_G f_T(x) dx. \end{aligned}$$

The estimate  $b(T) \leq \int_G f_T \leq b(T + e^{-T})$  leads to

$$1 \leq \frac{\int_G f_T}{b(T)} \leq \frac{b(T + e^{-T})}{b(T)} \leq \frac{b(T + \varepsilon)}{b(T)}$$

for every given  $\varepsilon > 0$ , if  $T \gg 0$ . As  $b$  is regular, we infer

$$\lambda_T \sim b(T), \quad \text{as } T \rightarrow \infty.$$

Let now  $U$  be an open neighborhood of the unit in  $G$  such that  $\gamma U \cap U = \emptyset$  for every  $\gamma \in \Gamma \setminus \{1\}$ . Let  $\eta \in C_c(\Gamma \backslash G)$  be of support inside  $\Gamma U$ , such that  $\eta \geq 0$  and  $\int_{\Gamma \backslash G} \eta(x) dx = 1$ . Write

$$\eta = \eta_{\text{const}} \oplus \eta_0,$$

where  $\eta_{\text{const}}$  stands for the projection onto the space of constant functions, i.e.,  $\eta_{\text{const}} = \langle \eta, \phi_{\text{const}} \rangle \phi_{\text{const}} = \frac{1}{\text{vol}(\Gamma \backslash G)} \int_{\Gamma \backslash G} \eta(x) dx = \frac{1}{\text{vol}(\Gamma \backslash G)}$ . Further,  $\eta_0 \in L_0^2(\Gamma \backslash G)$ . Then

$$R(f_T)\eta = \frac{\lambda_T}{\text{vol}(\Gamma \backslash G)} \oplus R(f_T)\eta_0$$

and so

$$\langle R(f_T)\eta, \eta \rangle = \frac{\lambda_T}{\text{vol}(\Gamma \backslash G)} + \langle R(f_T)\eta_0, \eta_0 \rangle$$

On the other hand, let  $\mathcal{F} \subset G$  be a measurable set of representatives for  $\Gamma \backslash G$ , where we can assume that  $U \subset \mathcal{F}$ . Then integrating a  $\Gamma$ -invariant function over  $\Gamma \backslash G$  means the same as integrating it over  $\mathcal{F}$  and so

$$\begin{aligned} \langle R(f_T)\eta, \eta \rangle &= \int_{\Gamma \backslash G} \int_{\Gamma \backslash G} k_T(x, y) \eta(y) dy \eta(x) dx \\ &= \int_{\mathcal{F}} \int_{\mathcal{F}} k_T(x, y) \eta(y) dy \eta(x) dx \\ &= \int_U \int_U k_T(x, y) \eta(y) dy \eta(x) dx \\ &= \sum_{\gamma \in \Gamma} \int_U \int_U f_T(x^{-1}\gamma y) \eta(x) \eta(y) dx dy. \end{aligned}$$

Now

$$\begin{aligned} f_T(x^{-1}\gamma y) &= \chi_T(d(x^{-1}\gamma yK, K)) \\ &= \chi_T(d(\gamma yK, xK)), \end{aligned}$$

and we have

$$\begin{aligned}
|d(\gamma yK, xK) - d(\gamma K, K)| &\leq |d(\gamma yK, xK) - d(\gamma yK, K)| \\
&\quad + |d(\gamma yK, K) - d(\gamma K, K)| \\
&\leq d(xK, K) + d(\gamma yK, \gamma K) \\
&= d(xK, K) + d(yK, K)
\end{aligned}$$

Let  $\varepsilon > 0$  and suppose that  $U$  is so small that for every  $x \in U$  we have  $d(xK, K) < \varepsilon/2$ . Then for any  $x, y \in U$  we have

$$d(\gamma xK, yK) - \varepsilon \leq d(\gamma K, K) \leq d(\gamma xK, yK) + \varepsilon.$$

As  $\chi_T$  is monotonically decreasing, it follows

$$\chi_T(d(\gamma xK, yK) + \varepsilon) \leq \chi_T(d(\gamma K, K)) \leq \chi_T(d(\gamma xK, yK) - \varepsilon).$$

By Lemma 1.5 we get for  $T$  large enough,

$$\chi_{T-\varepsilon}(d(\gamma xK, yK)) \leq \chi_T(d(\gamma K, K)) \leq \chi_{T+\varepsilon}(d(\gamma xK, yK)).$$

Integrating and summing we get for  $T \gg 0$ ,

$$\langle R(f_{T-\varepsilon})\eta, \eta \rangle \leq N(T) \leq \langle R(f_{T+\varepsilon})\eta, \eta \rangle.$$

In order to show the proposition, we need to estimate the term

$$\langle R(f_{T\pm\varepsilon})\eta_0, \eta_0 \rangle = \langle R(f_{T\pm\varepsilon})P\eta_0, P\eta_0 \rangle,$$

where  $P$  is the orthogonal projection to the set of  $K$ -invariants. The  $K$ -mixing property implies that the matrix coefficient  $x \mapsto \langle R(x)P\eta_0, P\eta_0 \rangle$  vanishes at infinity. So let  $\delta > 0$ , then there exists a compact set  $C \subset G$  such that  $|\langle R(x)P\eta_0, P\eta_0 \rangle| < \delta$  for  $x \notin C$ . Let  $G_T$  denote the set of all  $g \in G$  such that  $f_T(x) = 1$ . As  $T \rightarrow \infty$ ,  $G_T$  absorbs any compact set, so we may choose  $T_0$  so large that  $C \subset G_{T_0-\varepsilon}$ . For  $T \geq T_0$  we have

$$\begin{aligned}
\frac{1}{b(T)} |\langle R(f_{T\pm\varepsilon})\eta_0, \eta_0 \rangle| &= \frac{1}{b(T)} \left| \int_G f_{T\pm\varepsilon}(x) \langle R(x)P\eta_0, P\eta_0 \rangle dx \right| \\
&\leq \frac{1}{b(T)} \int_G f_{T\pm\varepsilon}(x) |\langle R(x)P\eta_0, P\eta_0 \rangle| dx \\
&\leq \frac{1}{b(T)} (b(T_0) + \delta b(T + \varepsilon)) \\
&= \frac{b(T_0)(1 - \delta)}{b(T)} + \delta \frac{b(T + \varepsilon)}{b(T)}.
\end{aligned}$$



The first summand tends to zero as  $T \rightarrow \infty$ . From this we conclude

$$\frac{N(T)}{b(T)} \leq \frac{|\lambda_T|}{b(T)} + \frac{b(T_0)(1-\delta)}{b(T)} + \delta \frac{b(T+\varepsilon)}{b(T)},$$

which implies that

$$\limsup_{T \rightarrow \infty} \frac{N(T)}{b(T)} = 1.$$

The limes inferior is dealt with analogously.  $\square$

**Corollary 1.6.** *Let the situation be as in the theorem, except that  $b(T)$  is not necessarily regular. Then we conclude that for every  $\varepsilon > 0$  we have, as  $T \rightarrow \infty$ ,*

$$b(T - \varepsilon) \ll N(T) \ll b(T + \varepsilon).$$

*Proof.* This follows from the proof of the theorem.  $\square$

**Definition 1.7.** By a *homogeneous metric measure space* we mean a quadruple  $(G, X, d, \mu)$  consisting of a group  $G$ , a non-compact proper metric space  $(X, d)$  on which  $G$  acts transitively and isometrically, and a  $G$ -invariant Radon measure  $\mu \neq 0$  on  $X$ .

**Definition 1.8.** The metric  $d$  is called an *inner metric* or *length metric*, if the distance  $d(x, y)$  equals the infimum of lengths of rectifiable curves joining  $x$  and  $y$ .

**Question 1.9.** *Let  $(G, X, d, \mu)$  be a homogeneous metric measure space and assume that the metric is inner. Let  $x_0 \in X$  and let*

$$b(T) = \mu \left( \left\{ x \in X : d(x, x_0) \leq T \right\} \right).$$

*Is it true that the function  $b$  is regular?*

**Example 1.10.** Let  $\mathcal{T}$  be a regular tree of valency  $q + 1$ ,  $q \in \mathbb{N}$ ,  $q \geq 2$  and let  $X$  be the set of vertices of  $\mathcal{T}$ , equipped with the path length metric of  $\mathcal{T}$ . The Automorphism group  $G$  of  $\mathcal{T}$  acts isometrically on  $X$  and with  $\mu$  being the counting measure,  $(G, X, d, \mu)$  is a homogeneous metric measure space. In this case we have for  $T \geq 1$ ,

$$b(T) = 1 + (q + 1)q^{[T]},$$

where  $[T]$  is the largest integer  $k \leq T$ . For  $n \in \mathbb{N}$ ,  $n \geq 3$  and  $0 < \varepsilon < 1$  we have

$$\frac{b(n - \varepsilon)}{b(n)} = \frac{1 + (q + 1)q^{n-2}}{1 + (q + 1)q^{n-1}} = \frac{1}{q} \frac{q + 1 + \frac{1}{q^{n-2}}}{q + 1 + \frac{1}{q^{n-1}}} \xrightarrow{n \rightarrow \infty} \frac{1}{q}.$$

Therefore, the function  $b(T)$  is not regular.

## 2 Small overlaps

Let  $(G, X, d, \mu)$  be a homogeneous metric measure space. Write

$$\overline{B}_T(x) = \{y \in X : d(x, y) \leq T\}$$

be the closed  $T$ -ball around  $x$ . For  $T, \delta > 0$  let  $[\overline{B}_T : \delta]$  be defined as

$$[\overline{B}_T : \delta] = \min \left\{ k \in \mathbb{N} : \overline{B}_T \subset \bigcup_{j=1}^k \overline{B}_\delta(x_j), \ x_1, \dots, x_k \in X \right\}.$$

For any  $T > 0$  write

$$b(T) = \mu(\overline{B}_T(x))$$

for any  $x \in X$  (doesn't depend on  $x$ ).

**Definition 2.1.** We say that  $X$  has *small overlaps*, if the metric  $d$  is inner and there exists a  $T_0 > 0$  such that

$$\frac{[\overline{B}_T : \delta] b(\delta)}{b(T)} \xrightarrow{\delta \rightarrow 0} 1$$

uniformly for  $T \in [T_0, \infty)$ .

**Example 2.2.** The euclidean metric on  $\mathbb{R}^n$ ,  $n \geq 2$  has no small overlaps. The metric on  $\mathbb{R}^n$  given by the 1-norm

$$\|x\|_1 = \sum_{j=1}^n |x_j| \quad \text{or} \quad \|x\|_\infty = \max_j |x_j|$$

both have small overlaps.

**Example 2.3.** Let  $\mathfrak{a}$  denote a finite-dimensional euclidean vector space and let  $\phi \subset \mathfrak{a}^* = \text{Hom}(\mathfrak{a}, \mathbb{R})$  be an irreducible root system. For  $\alpha \in \Phi$  and  $k \in \mathbb{N}$  define the affine hyperplane

$$H_{\alpha,k} = \{x \in \mathfrak{a} : \alpha(x) = k\}.$$

The reflections at all  $H_{\alpha,k}$  generate a group of affine isometries, the *Weyl group*  $W$ . The set  $\mathfrak{a} \setminus \bigcup_{\alpha,k} H_{\alpha,k}$  is a union of open simplices, called *affine chambers*. Let  $\overline{B}_1$  denote the closure of the union of those chambers  $C$  which have zero in their closure. Then  $\overline{B}_1$  is a closed convex neighborhood of zero, so there exists exactly one norm  $\|\cdot\|$  on  $\mathfrak{a}$  such that  $\overline{B}_1 = \{v \in \mathfrak{a} : \|v\| \leq 1\}$ . The metric on  $\mathfrak{a}$ , given by this norm has small overlaps.

**Proposition 2.4.** *If  $X$  is inner and has small overlaps, then the function  $b(T)$  is regular.*

*Proof.* Let  $\delta, \varepsilon, \tau > 0$ . Cover  $\overline{B}_T(x_0)$  with balls of radius  $\delta$  around the midpoints  $x_1, \dots, x_k$ . As the metric is inner, the balls of radius  $\delta + \varepsilon + \tau$  around  $x_1, \dots, x_k$  cover  $\overline{B}_{T+\varepsilon}(x_0)$ . So we have

$$\frac{b(T+\varepsilon)}{b(T)} \leq \frac{[\overline{B}_T : \delta]b(\delta + \varepsilon + \tau)}{b(T)}.$$

Let  $\alpha > 0$  be given and choose  $\delta > 0$  so small that for every  $T \geq T_0$  one has

$$\left| \frac{[\overline{B}_T : \delta]b(\delta)}{b(T)} - 1 \right| < \alpha.$$

Then for  $T \geq T_0$  we get

$$\frac{b(T+\varepsilon)}{b(T)} \leq (1+\alpha) \frac{b(\delta + \varepsilon + \tau)}{b(\delta)},$$

so that

$$1 \leq \lim_{\varepsilon \rightarrow 0} \limsup \frac{b(T+\varepsilon)}{b(T)} \leq (1+\alpha) \frac{b(\delta + \tau)}{b(\delta)},$$

and as  $\alpha$  and  $\tau$  are arbitrary, we get that the limit equals 1. Analogously one gets  $\lim_{\varepsilon \rightarrow 0} \liminf_{T \rightarrow \infty} \frac{b(T-\varepsilon)}{b(T)} = 1$ .  $\square$

The following lemma is of importance to our calculations.

**Lemma 2.5** (Persistence of the dominant asymptotic). *Let  $\mu, \nu$  be locally finite measures on  $[0, \infty)$  such that, for some  $\alpha \geq 0$  and some  $\beta > 0$ , as  $T \rightarrow \infty$ , one has*

$$\nu([0, T]) \sim T^\alpha e^{\beta T} \quad \text{and} \quad C = \int_{0, \infty)} e^{-\beta x} d\mu(x) < \infty.$$

Let

$$D(T) = \{(x, y) \in \mathbb{R}^2 : 0 \leq x, y \leq x + y \leq T\}.$$

Then, as  $T \rightarrow \infty$ , we have

$$\int_{D(T)} 1 d(\mu \otimes \nu) \sim C T^\alpha e^{\beta T}.$$

*Proof.* First note that the assumptions imply that  $\mu([0, T])e^{-\beta T}$  tends to zero as  $T \rightarrow \infty$ . This is seen as follows: Let  $0 < S < T$ , then

$$\begin{aligned} \mu([0, T])e^{-\beta T} &= \int_{[0, T]} \frac{e^{-\beta t}}{e^{\beta T - t}} d\mu(t) \\ &\leq \frac{1}{e^{\beta(T-S)}} \int_{[0, S]} e^{-\beta t} d\mu(t) + \int_{(S, T]} e^{-\beta t} d\mu(t). \end{aligned}$$

This implies that

$$\limsup_{T \rightarrow \infty} \mu([0, T])e^{-\beta T} \leq \int_{(S, \infty)} e^{-\beta t} d\mu(t).$$

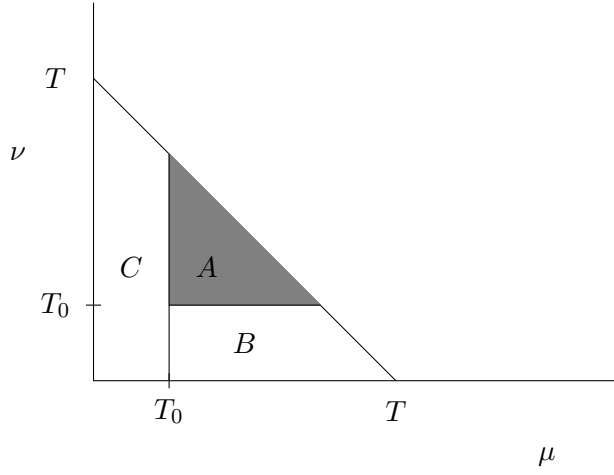
As the left hand side does not depend on  $S$ , it is zero. Next write  $d(T) = \int_{D(T)} d(\mu \otimes \nu)$ . Let  $T_0 > 0$  be so large that for every  $T \geq T_0$  one has

$$\left| \frac{\nu([0, T])}{T^\alpha e^{\beta T}} - 1 \right| < \varepsilon.$$

For  $T > 2T_0$  we have

$$\begin{aligned}
 \frac{d(T)}{T^\alpha e^{\beta T}} &= \int_{[0,T]} \frac{\nu([0, T-t])}{T^\alpha e^{\beta T}} d\mu(t) \\
 &= \int_{[0,T_0]} \frac{\nu([0, T-t])}{T^\alpha e^{\beta T}} d\mu(t) + \int_{(T_0,T]} \frac{\nu([0, T-t])}{T^\alpha e^{\beta T}} d\mu(t) \\
 &= \underbrace{\int_{[0,T_0]} \frac{\nu([0, T-t])}{T^\alpha e^{\beta T}} d\mu(t)}_{=a(T,T_0)} + \underbrace{\int_{(T_0,T]} \frac{\nu([0, T_0])}{T^\alpha e^{\beta T}} d\mu(t)}_{=b(T,T_0)} \\
 &\quad + \underbrace{\int_{(T_0,T]} \frac{\nu((T_0, T-t])}{T^\alpha e^{\beta T}} d\mu(t)}_{=c(T,T_0)}.
 \end{aligned}$$

These three summands are obtained by integrating  $\mu \otimes \nu$  over the areas  $A, B$  and  $C$  as in the following picture, which is why we call them  $a, b, c$ .



The summand  $b(T, T_0)$  tends to zero as  $T \rightarrow \infty$  by the first part of the proof.

The first summand is

$$\begin{aligned}
a(T, T_0) &= \int_{[0, T_0]} \frac{\nu([0, T-t])}{(T-t)^\alpha e^{\beta(T-t)}} \frac{(T-t)^\alpha e^{\beta(T-t)}}{T^\alpha e^{\beta T}} d\mu(t) \\
&\leq (1+\varepsilon) \int_{[0, T_0]} \frac{(T-t)^\alpha e^{\beta(T-t)}}{T^\alpha e^{\beta T}} d\mu(t) \\
&= (1+\varepsilon) \int_{[0, T_0]} e^{-\beta t} \frac{(T-t)^\alpha}{T^\alpha} d\mu(t) \\
&\leq (1+\varepsilon) \int_{[0, T_0]} e^{-\beta t} d\mu(t),
\end{aligned}$$

so that

$$\limsup_{T \rightarrow \infty} a(T, T_0) \leq (1+\varepsilon) \int_{[0, T_0]} e^{-\beta t} d\mu(t).$$

On the other hand we have that

$$\begin{aligned}
a(T, T_0) &\geq (1-\varepsilon) \int_{[0, T_0]} e^{-\beta t} \frac{(T-t)^\alpha}{T^\alpha} d\mu(t) \\
&\geq (1-\varepsilon) \int_{[0, T_0]} e^{-\beta t} d\mu(t) \frac{(T-T_0)^\alpha}{T^\alpha}
\end{aligned}$$

so that

$$\liminf_{T \rightarrow \infty} a(T, T_0) \geq (1-\varepsilon) \int_{[0, T_0]} e^{-\beta t} d\mu(t).$$

As for the third summand we have

$$\begin{aligned}
c(T, T_0) &= \int_{(T_0, T-T_0]} \frac{\nu((T_0, T-t])}{T^\alpha e^{\beta T}} d\mu(t) \\
&\leq \int_{(T_0, T-T_0]} \frac{\nu([0, T-t])}{T^\alpha e^{\beta T}} d\mu(t) \\
&= \int_{(T_0, T-T_0]} e^{-\beta t} \frac{\nu([0, T-t])}{(T-t)^\alpha e^{\beta(T-t)}} \frac{(T-t)^\alpha}{T^\alpha} d\mu(t) \\
&\leq (1+\varepsilon) \int_{(T_0, T-T_0]} e^{-\beta t} \frac{(T-t)^\alpha}{T^\alpha} d\mu(t) \\
&\leq (1+\varepsilon) \int_{[T_0, \infty)} e^{-\beta t} d\mu(t).
\end{aligned}$$

Summing up, we get

$$\begin{aligned} (1 - \varepsilon) \int_{[0, T_0]} e^{-\beta t} d\mu(t) &\leq \liminf_{T \rightarrow \infty} \frac{d(T)}{T^\alpha e^{\beta T}} \\ &\leq \limsup_{T \rightarrow \infty} \frac{d(T)}{T^\alpha e^{\beta T}} \leq (1 + \varepsilon) \int_{[0, \infty)} e^{-\beta t} d\mu(t) \end{aligned}$$

Letting  $\varepsilon$  tend to zero and  $T_0$  to infinity accordingly, we get

$$\lim_{T \rightarrow \infty} \frac{d(T)}{T^\alpha e^{\beta T}} = \int_{[0, \infty)} e^{-\beta t} d\mu(t).$$

The lemma follows.  $\square$

**Corollary 2.6.** *Let the setting be as in the lemma, except that we only have*

$$\nu([0, T]) \ll T^\alpha e^{\beta T}$$

*as  $T \rightarrow \infty$ . Then we get*

$$\int_{D(T)} 1 d(\mu \otimes \nu) \ll C T^\alpha e^{\beta T}$$

*and the same for  $\gg$ .*

*Proof.* Inspection of the proof of the lemma.  $\square$

### 3 The adelic metric space

**Definition 3.1.** Let  $\mathbb{G}$  denote a semisimple linear algebraic group over  $\mathbb{Q}$ . For a ring extension  $R/\mathbb{Q}$  we write  $\mathbb{G}_R$  for the group of  $R$ -valued points. By  $p$  we denote either a prime number, in which case we write  $p < \infty$  or  $p = \infty$ . In both cases we say that  $p$  is a *place* of  $\mathbb{Q}$ . We write  $\mathbb{Q}_p$  for the completion of  $\mathbb{Q}$  at  $p$ , so for instance  $\mathbb{Q}_\infty = \mathbb{R}$ .

**Definition 3.2.** Let  $\mathbb{A} = \mathbb{A}_{\text{fin}} \times \mathbb{A}_\infty$  denote the adèle ring of  $\mathbb{Q}$ , where  $\mathbb{A}_\infty = \mathbb{R}$  and  $\mathbb{A}_{\text{fin}}$  is the ring of finite adeles, i.e., the restricted product of all  $\mathbb{Q}_p$  with  $p < \infty$ . Writing  $\mathbb{G}_{\text{fin}} = \mathbb{G}_{\mathbb{A}_{\text{fin}}}$  and  $\mathbb{G}_\infty = \mathbb{G}_{\mathbb{R}}$ , we have  $G = \mathbb{G}_{\mathbb{A}} = \mathbb{G}_{\text{fin}} \times \mathbb{G}_\infty$ . We abbreviate  $\mathbb{G}_{\mathbb{Q}_p}$  by  $\mathbb{G}_p$ . Fix an embedding  $\mathbb{G} \hookrightarrow \text{GL}_n$  over  $\mathbb{Q}$ . For every  $p < \infty$  set  $\mathbb{G}_{\mathbb{Z}_p} = \mathbb{G}_p \cap \text{GL}_n(\mathbb{Z}_p)$ . For each  $p \leq \infty$  we fix a maximal compact subgroup  $K_p$  of  $\mathbb{G}_p$  in such a way that

- if  $p < \infty$ , then  $K_p$  is special, i.e., it fixes a special vertex in the Bruhat-Tits building,
- for almost all  $p$ , the group  $K_p$  is hyperspecial and we have  $K_p = \mathbb{G}_{\mathbb{Z}_p}$ .

For the possibility of such a choice, see [Tit79, 3.9.1]. We need to fix Haar measures. For every  $p \leq \infty$  we normalize the Haar measure on  $K_p$  by insisting that  $\text{vol}(K_p) = 1$ . For  $p < \infty$  this already fixes the Haar measure on  $G_p$ . On  $\mathbb{G}_\infty$  any Haar measure is induced from a  $\mathbb{G}_\infty$ -invariant Riemannian metric on the symmetric space  $X_\infty = \mathbb{G}_\infty/K_\infty$ , which we choose to be the one given by the Killing form  $b$ .

**Definition 3.3.** For every  $p < \infty$  we denote by  $\mathcal{B}_p$  the Bruhat-Tits building of the group  $\mathbb{G}_p$ . The building  $\mathcal{B}_p$  carries a metric which is euclidean on each apartment. For arithmetic purposes it has turned out that, as far as only vertices of the building are concerned, the combinatorial distance is more useful, see [Man91, Wer02]. This is a metric on the set  $\mathcal{B}_{p,0}$  of all vertices of the building defined by considering the graph  $\mathcal{B}_{p,1}$ , which is the 1-skeleton of  $\mathcal{B}_p$ . For two vertices  $x, y$  of  $\mathcal{B}_p$  let  $d_p(x, y)$  denote the combinatorial distance in  $\mathcal{B}_{p,1}$ , i.e., the length of the shortest path joining  $p$  and  $q$ , where each edge gets the length 1. Note that the group  $K_p$  fixes a unique vertex  $o_p$  in  $\mathcal{B}_p$  and in this way we get a canonical injection  $\mathbb{G}_p/K_p \hookrightarrow \mathcal{B}_{p,0}$ ,  $xK_p \mapsto xo_p$ . We will denote the point  $xo_p \in \mathcal{B}_p$  also by  $xK_p$ .

**Definition 3.4.** Also for  $p = \infty$ , there is a metric on the symmetric space  $X_\infty = \mathbb{G}_\infty/K_\infty$ . However, as at the finite places, it turns out, that for our purposes a different metric is more useful. First observe that for given choice of a maximal compact subgroup  $K_\infty$ , there exists a maximal connected  $\mathbb{R}$ -split torus  $A$  in  $G$  such that  $\theta(a) = a^{-1}$  for every  $a \in A$ , where  $\theta$  is the Cartan involution on  $G$ , fixing  $K_\infty$  pointwise. Let  $\mathfrak{a}$  be the Lie algebra of  $A$  and let  $\Phi \subset \mathfrak{a}^*$  be the set of roots of  $(\mathfrak{a}, \mathfrak{g})$ , where  $\mathfrak{g}$  is the Lie algebra of  $\mathbb{G}_\infty$ . For each  $\alpha \in \Phi$  let  $m_\alpha$  denote the dimension of the corresponding root space  $\mathfrak{g}_\alpha$ . Fix a choice of positive roots  $\Phi^+ \subset \Phi$  and let  $\rho = \frac{1}{2} \sum_{\alpha \in \Phi^+} m_\alpha \alpha$  be the modular shift. Choose some  $B > 0$  and set

$$\|X\|_B = \frac{1}{2B} \sum_{\alpha \in \Phi^+} m_\alpha |\alpha(X)|.$$

Then  $\|\cdot\|_B$  is a Weyl group invariant norm on  $\mathfrak{a}$ . For  $X \in \mathfrak{a}^+$ , the positive Weyl chamber given by the ordering, we get  $\|X\|_B = \frac{\rho(X)}{B}$ .



We define a  $\mathbb{G}_\infty$ -invariant metric  $d_\infty$  on  $X_\infty$  by setting

$$d_\infty(xK_\infty, yK_\infty) = \tilde{d}_\infty(y^{-1}xK),$$

where the one argument function  $\tilde{d}_\infty$  is defined by

$$\tilde{d}_\infty(K \exp(X)K) = \|X\|_B, \quad X \in \mathfrak{a}.$$

**Definition 3.5.** On the group  $\mathbb{G}_p$ ,  $p \leq \infty$ , we now define the local height  $h_p : \mathbb{G}_p \rightarrow [1, \infty)$  by the formula

$$h_p(x) = \begin{cases} p^{d_p(xK_p, K_p)} & p < \infty, \\ e^{d_\infty(xK_\infty, K_\infty)} & p = \infty. \end{cases}$$

The global height on  $\mathbb{G}_\mathbb{A}$  is defined by

$$h(x) = \prod_{p \leq \infty} h_p(x).$$

We denote the corresponding counting function by  $\pi$ . So

$$\pi(x) = \#\{\gamma \in \mathbb{G}_\mathbb{Q} : h(\gamma) \leq x\}.$$

Let  $X_\infty$  be the symmetric space  $\mathbb{G}_\infty/K_\infty$  and for  $p < \infty$  set  $X_p = \mathcal{B}_{p,1}$ . Let

$$X = \bigoplus_{p \leq \infty} X_p$$

denote the set of all  $x \in \prod_{p \leq \infty} X_p$  such that  $x_p = K_p$  holds for almost all  $p$ . Let  $x \in X$  then the stabilizer  $K_x$  of  $x$  in  $\mathbb{G}_\mathbb{A}$  is a maximal compact subgroup of  $\mathbb{G}_\mathbb{A}$  which stabilizes no other point, hence the map  $x \mapsto K_x$  is an injection of  $X$  into the set of all maximal compact subgroups of  $\mathbb{G}_\mathbb{A}$ . We occasionally denote elements of  $X$  as compact subgroups of  $\mathbb{G}_\mathbb{A}$ . In particular, the group  $K_\mathbb{A} = \prod_{p \leq \infty} K_p$  defines a base-point in  $X$ . We can identify  $X$  with  $\mathbb{G}_\mathbb{A}/K_\mathbb{A}$ .

**Proposition 3.6.** *For  $p = \infty$  we formally write  $\log p = \log e = 1$ . The sum*

$$d(x, y) = \sum_{p \leq \infty} d_p(x_p, y_p) \log p$$

*is finite for each given pair  $x, y \in X$  and defines a metric on  $X$  which makes  $X$  a complete and proper metric space. The natural action of  $\mathbb{G}_\mathbb{A}$  in  $X$  is isometric and the metric topology equals the topology induced by  $X \cong \mathbb{G}_\mathbb{A}/K_\mathbb{A}$ . Further, the  $\mathbb{G}_\infty$ -invariant measure on  $\mathbb{G}_\infty/K_\infty$  and the counting measures on the  $X_p$  for  $p < \infty$  give a  $G$ -invariant, non-trivial Radon measure on  $X$ , which coincides with the quotient of the Haar measures on  $\mathbb{G}_\mathbb{A}$  and  $K_\mathbb{A}$ .*

*Proof.* To show completeness, let  $(x_i)$  be a Cauchy sequence and let  $i_0 \in \mathbb{N}$  be such that for all  $i, j \geq i_0$  we have  $d(x_i, x_j) < \log 2$ . Then for any given  $p < \infty$  we have

$$d_p(x_{i,p}, x_{j,p}) \log p \leq \sum_{q \leq \infty} d_q(x_{i,q}, x_{j,q}) \log q = d(x_i, x_j) < \log 2$$

So  $d_p(x_{i,p}, x_{j,p}) < \frac{\log 2}{\log p} \leq 1$  and therefore  $d_p(x_{i,p}, x_{j,p}) = 0$ , which amounts to  $x_{i,p} = x_{j,p}$  for all  $i, j \geq i_0$ . Since this holds for every prime  $p < \infty$ , the completeness follows from the completeness of the metric space  $X_\infty$ .

For properness, it suffices to show that every closed Ball  $B_R(K)$  for given  $R > 1$  is compact. For this let  $P$  denote the set of all places  $p$  such that  $\log p \leq R$ . Then the set  $P$  is finite. Let  $x \in B_R(K)$ , then for each  $p \notin P$  we have

$$d_p(x_p K_p, K_p) \log p \leq d(xK, K) \leq R.$$

Therefore  $d_p(x_p K_p, K_p) < 1$  hence it is zero. So the ball  $B_R(K)$  is homoeomorphic to the corresponding ball in  $\prod_{v \in P} X_p$  which is compact. The rest of the lemma is clear.  $\square$

## 4 Lattice points of $\mathbb{Q}$ -groups

We make the following assumptions.

- (a) We assume that  $\mathbb{G}$  is connected and almost  $\mathbb{Q}$ -simple.
- (b) We assume  $\mathbb{G}_{\mathbb{R}}$  is not compact and the group homomorphism  $\tilde{\mathbb{G}}_{\mathbb{A}} \rightarrow \mathbb{G}_{\mathbb{A}}$  is surjective, where  $\tilde{\mathbb{G}}$  is the simply connected covering group of  $\mathbb{G}$ .

The second part of condition (b) is satisfied if  $\mathbb{G}$  is itself simply connected, but also in some other cases like  $\mathbb{G} = \mathrm{PGL}_n$ , see Section 5.

**Definition 4.1.** By an *automorphic character* of  $\mathbb{G}$  we mean a continuous group homomorphism  $\chi : G = \mathbb{G}_{\mathbb{A}} \rightarrow \mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$  such that  $\chi(\Gamma) = 1$ , where  $\Gamma = \mathbb{G}_{\mathbb{Q}}$ . Since  $\mathrm{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}}) < \infty$ , it follows that  $\chi \in L^2(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})$ . Let  $L^2_{00}(\Gamma \backslash G)$  denote the orthogonal space of all automorphic characters.

We formulate another condition

(b') With  $G = \mathbb{G}_{\mathbb{A}}$  and  $K = K_{\mathbb{A}}$  we have

$$L_0^2(\Gamma \backslash G)^K \subset L_{00}^2(\Gamma \backslash G).$$

**Lemma 4.2.** *If  $\mathbb{G}$  satisfies (a) and (b), then it satisfies (b').*

*Proof.* Suppose we know the claim in the case  $\tilde{\mathbb{G}} = \mathbb{G}$ . If  $\tilde{\mathbb{G}}_{\mathbb{A}} \rightarrow \mathbb{G}_{\mathbb{A}}$  is onto, then every automorphic character of  $\mathbb{G}_{\mathbb{A}}$  induces one of  $\tilde{\mathbb{G}}_{\mathbb{A}}$  and so one sees that the space  $L_0^2(\Gamma \backslash G)$  equals the image of  $L_0^2(\tilde{\Gamma} \backslash \tilde{G})$  and the same holds for  $L_{00}^2$  so if  $L_0^2(\tilde{\Gamma} \backslash \tilde{G})^K \subset L_{00}^2(\tilde{\Gamma} \backslash \tilde{G})$ , the same follows for  $G$ . So assume that  $\mathbb{G}$  is simply connected. By strong approximation [Rap14] one then has  $\mathbb{G}_{\mathbb{A}} = \mathbb{G}_{\mathbb{Q}} \mathbb{G}_{\mathbb{R}} K$ . So if  $\chi$  is a  $K$ -invariant automorphic character, then  $\chi(\mathbb{G}_{\mathbb{A}}) = \chi(\mathbb{G}_{\mathbb{R}})$ . By Lemma 4.7 of [GMO08] we have  $\chi(\mathbb{G}_{\mathbb{R}}^0) = 1$  and  $K \cap \mathbb{G}_{\mathbb{R}}$  meets every connected component of  $\mathbb{G}_{\mathbb{R}}$ , hence  $\chi$  is trivial in this case.  $\square$

**Lemma 4.3.** *Let  $\mathbb{G}$  satisfy the conditions (a) and (b'). Let  $G = \mathbb{G}_{\mathbb{A}}$  as well as  $\Gamma = \mathbb{G}_{\mathbb{Q}}$  and  $K = K_{\mathbb{A}}$ . Then the group  $G$  acts  $K$ -mixingly on  $\Gamma \backslash G$ .*

*Proof.* Let  $L_{00}^2(\Gamma \backslash G)$  denote the orthogonal space of all automorphic characters and let  $\psi$  be a  $K$ -invariant matrix coefficient. By condition (b') we can write  $\psi = \psi_{v,w}$  for some  $v, w \in L_{00}^2$  and then the claim follows from [GMO08, Theorem 3.26].  $\square$

**Definition 4.4.** Let  $\mathbb{G}$  be a semisimple linear algebraic group. For a prime number  $p$  and  $k \in \mathbb{N}_0$  let  $D(p^k)$  denote the number of vertices  $v$  of  $\mathcal{B}_p$  in the  $G_p$ -orbit of  $K_p$  such that the distance  $d_p(v, K_p)$  is  $k$ . For a natural number  $m$  with prime decomposition  $m = p_1^{k_1} \cdots p_t^{k_t}$  let

$$D(m) = D(p_1^{k_1}) \cdots D(p_t^{k_t}).$$

**Lemma 4.5.** *Let  $\mathbb{G}$  be a semisimple linear algebraic group over  $\mathbb{Q}$  of absolute rank  $r$ . Then the Dirichlet series*

$$L(s) = \sum_{m=1}^{\infty} \frac{D(m)}{m^s}$$

*has a finite abscissa of convergence  $B_0 < \infty$ .*

*Proof.* Embedding the group into some  $\mathrm{SL}_n$  and using the corresponding bounds to the Bruhat-Tits building of  $\mathrm{SL}_n$  (Lemma 5.2) we deduce that there exists a number  $n \in \mathbb{N}$  and  $C > 0$  such that

$$D(p) \leq Cp^n.$$

As every vertex in the orbit of  $K_p$  has  $D(p)$  orbits, we deduce

$$D(p^k) \leq C^k D(p)^k$$

and hence

$$D(m) \leq C^{k_1 + \dots + k_t} m^n,$$

if  $m$  has prime decomposition  $m = p_1^{k_1} \dots p_t^{k_t}$ . But then  $k_1 + \dots + k_t \leq \frac{\log m}{\log 2}$ , so that

$$D(m) \leq m^{n + \frac{\log C}{\log 2}}.$$

The lemma follows.  $\square$

**Theorem 4.6.** *Let  $\mathbb{G}$  satisfy the conditions (a) and (b) of Section 4. Let  $r$  denote the absolute rank of  $\mathbb{G}$  and  $r_{\mathbb{R}}$  its rank over  $\mathbb{R}$ . Let  $B_0 > 0$  as in the last lemma and suppose that  $B > B_0$ . Then the counting function  $N(T)$  of the group  $\mathbb{G}$  satisfies*

$$N(T) \sim \frac{aC}{\text{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} (BT)^{r_{\mathbb{R}}-1} e^{BT},$$

where  $a$  is the area of an explicitly given  $r_{\mathbb{R}} - 1$ -dimensional simplex and  $C = L(B) = \sum_{m=1}^{\infty} \frac{D(m)}{m^B}$ .

*Proof.* Let  $A$  be a maximal connected split torus in  $G_{\infty} = \mathbb{G}(\mathbb{R})$ . Setting  $b^{\infty}(R) = \text{vol}(B_R^{\infty})$  we use the Cartan integral formula [Kna86, p.142] to see that  $b^{\infty}(R)$  equals

$$\int_{\mathfrak{a}_R^+} \prod_{\alpha > 0} \left( \frac{e^{\alpha(X)} - e^{-\alpha(X)}}{2} \right)^{m_{\alpha}} dX,$$

where  $\mathfrak{a}_R^+$  is the set of all  $X \in \mathfrak{a}^+$  that satisfy  $\rho(X) \leq BR$ .

The leading term of  $b^{\infty}(R)$  in the asymptotic as  $R \rightarrow \infty$  is

$$\int_{\mathfrak{a}_R^+} \prod_{\alpha > 0} \left( \frac{e^{\alpha(X)}}{2} \right)^{m_{\alpha}} da = 2^{-\sum_{\alpha > 0} m_{\alpha}} \int_{\mathfrak{a}_R^+} e^{2\rho(X)} da$$

Setting  $y = \rho(X)$  we find that this equals

$$\frac{1}{2^{\sum_{\alpha > 0} m_{\alpha}}} \int_0^{BR} F(y) e^y dy,$$

where  $F(y)$  equals the area of the  $(r_{\mathbb{R}} - 1)$ -dimensional submanifold of  $\mathfrak{a}$  given as  $\mathfrak{a}_R^+ \cap \{\rho(X) = y\}$ . Here we use that  $\mathfrak{a}$  carries a euclidean structure inducing the Haar measure. This structure is not unique, but when we determine the covector  $\rho$  to be of length 1, the induced measure on the orthocomplement  $\{X : \rho(X) = 0\}$  is uniquely determined. In this way,  $F(y)$  is well-defined. As the equations/inequalities describing this manifold are all linear, we infer that  $F(y) = a^{d-2}$  for  $\alpha_d = F(1)$  which is the volume of a certain  $(r_{\mathbb{R}} - 1)$ -dimensional simplex in  $\mathfrak{a}$ .

It follows that, as  $R \rightarrow \infty$ ,

$$b^\infty(R) \sim a(BR)^{r_{\mathbb{R}}-1} e^{BR}.$$

Now the Theorem follows from Lemma 2.5 and Theorem 1.4.  $\square$

## 5 The group $\mathrm{PGL}(d)$

We want to make sure that the conditions of Section 4 are satisfied for the group  $\mathrm{PGL}_d$  with  $d \geq 2$ . The conditions (a) is clearly satisfied. Condition (b) follows from the next lemma.

**Lemma 5.1.** *Let  $R$  be a ring and  $2 \leq d \in \mathbb{N}$ . Then  $\mathrm{PGL}_d(R) = \mathrm{GL}_d(R)/R^\times$  holds for  $R$  being a field or  $R = \mathbb{Z}_p$  as well as  $R = \mathbb{A}_S$  for any set of places  $S$ .*

*Proof.* The sequence of algebraic groups

$$1 \rightarrow \mathrm{GL}_1 \rightarrow \mathrm{GL}_d \rightarrow \mathrm{PGL}_d \rightarrow 1$$

is exact. By Hilbert's Theorem 90, for each field  $k$  the Galois cohomology  $H^1(k, \mathrm{GL}_1)$  is trivial and so the exact sequence of Galois cohomology shows that the sequence

$$1 \rightarrow \mathrm{GL}_1(k) \rightarrow \mathrm{GL}_d(k) \rightarrow \mathrm{PGL}_d(k) \rightarrow 1$$

is exact, which implies the claim for fields. To verify the claim for  $R = \mathbb{Z}_p$ , we have to analyze the coordinate ring of  $\mathrm{PGL}_d$ . First, the coordinate ring of  $\mathrm{GL}_d$  over  $R$  is

$$\mathcal{O}_{\mathrm{GL}_d} = R[x_{ij}, y] / \det(x)y - 1.$$

The coordinate ring of  $PGL_d = GL_d / GL_1$  is the ring of  $GL_1$ -invariants, where the action of  $GL_1$  is given by

$$\lambda.f(x_{ij}, y) = f(\lambda x_{ij}, \lambda^{-d} y).$$

The ring of invariants is generated by all monomials of the form  $x_{i_1 j_1} \cdots x_{i_d j_d} y$  for  $1 \leq i_k, j_k \leq d$ . Let now  $\chi \in PGL_d(\mathbb{Z}_p)$ . Then  $\chi$  is a homomorphism from  $\mathcal{O}_{PGL_d}$  to  $\mathbb{Z}_p$ . Every such can be extended to  $\mathcal{O}_{GL_d} \rightarrow \mathbb{Q}_p$  and we have to show that there exists an extension mapping to  $\mathbb{Z}_p$ . Pick any extension and denote it by the same letter  $\chi$ . For the valuation  $v$  on  $\mathbb{Q}_p$  we have

$$0 \leq v(\chi(x_{ij}^d y)) = dv(\chi(x_{ij})) + v(\chi(y)).$$

We are free to change  $\chi(y)$  to  $\chi(y)p^{dk}$  for any  $k \in \mathbb{Z}$  if at the same time we change  $\chi(x_{ij})$  to  $\chi(x_{ij})p^{-k}$ . Thus we can assume  $v(\chi(y)) \in \{0, 1, \dots, d-1\}$ . Then we conclude  $v(\chi(x_{ij})) \geq 0$  for all  $i, j$  and so  $\chi$  indeed maps into  $\mathbb{Z}_p$  as claimed. This shows the result for  $\mathbb{Z}_p$ . The ring  $\mathbb{A}_S$  is the restricted product of the fields  $\mathbb{Q}_p$ ,  $p \in S$  with respect to the compact subrings  $\mathbb{Z}_p$ . So the result for  $\mathbb{Q}_p$  and  $\mathbb{Z}_p$  implies the result for  $\mathbb{A}_S$ .  $\square$

We want to determine the asymptotic growth of  $b(T)$  in the case  $\mathbb{G} = PGL_d$ . By definition we have

$$b(T) = \sum_{m \leq e^T} \left[ \prod_{p^k || m} \text{vol}(D_k^p) \right] \text{vol}(B_{T-\log m}^\infty).$$

Here the sum runs over all  $m \in \mathbb{N}$  with  $m \leq e^T$ , the product extends over all prime numbers  $p$ , where  $p^k$  is the exact power of  $p$  dividing  $m$ . Further  $\text{vol}(D_k^p)$  is the volume (=cardinality) of the set

$$D_k^p = \left\{ x \in X_p : d_p(x, K_p) = k \right\}.$$

Note that  $D_k^p = B_k^p \setminus B_{k-1}^p$ , where  $B_k^p$  is the closed ball of radius  $k$  in  $X_p$ . As  $\text{vol}(D_0^p) = 1$ , the product above is finite.

For  $m \in \mathbb{N}$  we write

$$D(m) = D_d(m) = \prod_{p^k || m} \text{vol}(D_k^p) = \prod_{p^k || m} D(p^k).$$

**Lemma 5.2.** *For every prime number  $p$  and every  $k \in \mathbb{N}$  we have*

$$D(p^k) = D(p)c(p)^{k-1}$$

where

$$c(p) = c_d(p) = (d-1)p^{d-1} + p^{d-2} + \cdots + p = (d-1)p^{d-1} + \frac{p^{d-1} - 1}{p - 1} - 1.$$

One has

$$D(p) = (d-1)(p^{d-1} + \cdots + 1) = (d-1)\frac{p^d - 1}{p - 1}.$$

Let  $s_p = s_p(d) = \frac{\log(c(p))}{\log p}$ . The series

$$L(s) = \sum_{m=1}^{\infty} \frac{D(m)}{m^s}$$

converges absolutely for  $\operatorname{Re}(s) > s_2(d)$  if  $d \geq 3$  and for  $\operatorname{Re}(s) > 2$  if  $d = 2$ . The function  $L(s)$  extends to a meromorphic function in  $\{\operatorname{Re}(s) > d\}$  with at most simple poles at

$$s_p + 2\pi i \log(p)k, \quad k \in \mathbb{Z}$$

where  $p$  is a prime number. As the prime  $p$  tends to infinity, the sequence of real parts of the poles,  $s_p = \frac{\log(c(p))}{\log p}$ , converges monotonically from above to  $d - 1$ , which means that only finitely many of them are  $> d$ . The first values for  $s_2$  are

$n$	$s_2$	$s_3$
2	1	1
3	3.3219280949	2.7712437492
4	4.9068905956	4.1257498573
5	6.2854022189	5.3653166773
6	7.5698556083	6.5507064185

In the special case  $d = 2$  we have

$$L(s) = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)},$$

where  $\zeta$  denotes the Riemann zeta function.

*Proof.* We show the equivalent formula for the ball volume

$$\text{vol}(B_k^p) = 1 + (d-1) \frac{p^d - 1}{p-1} \left[ \frac{\left( (d-1) \frac{p^d-1}{p-1} - (d-2) \frac{p^{d-1}-1}{p-1} - 1 \right)^k - 1}{(d-1) \frac{p^d-1}{p-1} - (d-2) \frac{p^{d-1}-1}{p-1} - 2} \right].$$

Note that the building  $\mathcal{B}_p$  actually also is the building for the group  $\text{SL}_d$ . As it is slightly more convenient here, we will use the latter group for the computations. We begin by establishing a formula for the number of chambers of  $\mathcal{B}_p$  incident with a fixed  $m$ -simplex  $\sigma \subset \mathcal{B}_p$  where  $m \leq d-2$ . This can be found by counting the number of minimal parahoric subgroups contained in the pointwise stabiliser of  $\sigma$  in  $G_p = \text{SL}_d(\mathbb{Q}_p)$ . The pointwise stabiliser of  $\sigma$  in  $G_p$  is itself a parahoric subgroup  $G_{p,\sigma}$ , and the number of minimal parahoric subgroups that it contains is equal to the number of conjugates of a fixed minimal parahoric subgroup  $G_{p,C} \subset G_{p,\sigma}$ , where  $C$  is some chamber incident to  $\sigma$  and  $G_{p,C}$  is its pointwise stabiliser in  $G_p$ , since the chambers incident with  $\sigma$  comprise a single  $G_{p,\sigma}$ -orbit. So the desired number may be computed by finding the index in  $G_{p,\sigma}$  of the normaliser in  $G_{p,\sigma}$  of  $G_{p,C}$ . But parahoric subgroups are self-normalising in  $G_p$ , and so for that reason the number in question is equal to the index of  $G_{p,C}$  in  $G_{p,\sigma}$ . The number does not depend on our choice of  $C$  and  $\sigma$ , so it is sufficient to calculate it in the special case where  $G_{p,C}$  is the subgroup of  $\text{SL}_n(\mathbb{Z}_p)$  such that the entries on or above the main diagonal lie in  $\mathbb{Z}_p$  and the entries below the main diagonal lie in the maximal ideal  $p\mathbb{Z}_p$ , and  $G_{p,\sigma}$  is some strictly larger parahoric subgroup. In this case the index of  $G_{p,C}$  in  $G_{p,\sigma}$  may be found by considering the projections to  $\text{SL}_d(\mathbb{F}_p)$ , and we can calculate it by counting flags of a fixed type in  $\mathbb{P}^{d-1}(k)$ .

Thus we obtain the result that the number of chambers of  $\mathcal{B}_p$  incident to a fixed  $m$ -simplex where  $m$  is an integer such that  $m \leq d-2$ , is given by  $\prod_{j=2}^{d-m} \frac{p^j-1}{p-1}$ . Now let us apply this result to counting the number  $b_k^p$  of vertices of distance at most  $k$  from a fixed vertex  $v$  in the 1-skeleton of  $\mathcal{B}_p$ . We clearly have  $b_0^p = 1$ . To calculate  $b_1^p$  we must count the number of edges incident to  $v$ . There are  $\prod_{j=2}^d \frac{p^j-1}{p-1}$  chambers incident to  $v$  and each one is incident with  $d-1$  edges which are incident with  $v$ . But each edge gets counted  $\prod_{j=2}^{d-1} \frac{p^j-1}{p-1}$  times, so that we obtain  $b_1^p = 1 + (d-1) \frac{p^d-1}{p-1}$ . To generalise this to the case  $k=2$  we must count how many vertices of distance 2 from  $v$  are joined by an edge to some fixed vertex  $w$  of distance 1 from  $v$ . There are  $(d-1) \frac{p^d-1}{p-1}$  vertices joined by an edge to  $w$  altogether, we must exclude  $v$  and vertices at distance 1 from  $v$ . The number of vertices at



distance 1 from  $v$  which are joined by an edge to  $w$  is equal to the number of edges incident to  $w$  which are incident to chambers incident to  $v$  but not to  $v$  itself. This number is  $(d-2)\frac{p^{d-1}-1}{p-1}$ . We have now justified the formula in the case  $k=2$  and at this point it is clear how to obtain the general result by induction.

The statement on the convergence of the Dirichlet series is obtained as follows. First, using the estimate  $k_1 + \dots + k_s \leq \frac{\log m}{\log 2}$ , if  $m = p_1^{k_1} \dots p_s^{k_s}$ , where the  $p_j$  are the different primes dividing  $m$ , we obtain  $D(m) \leq m^{d-1+\frac{\log d}{\log 2}}$ , so that the Dirichlet series converges absolutely for  $\text{Re}(s) \gg 0$ . As the coefficient  $D(m)$  is weakly multiplicative, in the domain of absolute convergence the series admits an Euler product of the form

$$\prod_p \left( 1 + \sum_{k=1}^{\infty} D(p^k) p^{-ks} \right).$$

Write

$$c(p) = (d-1)p^{d-1} + \frac{p^d - 1}{p - 1} - 1.$$

For  $k \in \mathbb{N}$  we have  $D(p^k) = D(p)c(p)^{k-1}$ , so that the Euler product becomes

$$\begin{aligned} \prod_p \left( 1 + \sum_{k=1}^{\infty} D(p)c(p)^{k-1} p^{-ks} \right) &= \prod_p \left( 1 + \frac{D(p)}{c(p)} \sum_{k=1}^{\infty} (c(p)p^{-s})^k \right) \\ &= \prod_p \left( 1 + \frac{D(p)}{c(p)} \frac{c(p)p^{-s}}{1 - c(p)p^{-s}} \right) \\ &= \prod_p \frac{1 - [c(p) - D(p)] p^{-s}}{1 - c(p)p^{-s}}. \end{aligned}$$

In the case  $d=2$  we have  $c(p) = p$  and  $D(p) = p+1$ , so that in this case,

$$L(s) = \prod_p \frac{1 + p^{-s}}{1 - p^{-(s-1)}} = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}.$$

In general, as  $c(p), D(p) \leq 2(d-1)p^{d-1}$  we infer that the Euler products

$$\prod_p (1 - c(p)p^{-s})$$

and

$$\prod_p (1 - [c(p) - D(p)] p^{-s})$$

both converge for  $\operatorname{Re}(s) > d$  thus extending the holomorphic function  $\sum_m \frac{D(m)}{m^s}$  to the range  $\operatorname{Re}(s) > d$  minus the set of zeros of the Euler factors  $(1 - c(p)p^{-s})$ . These zeros are exactly at the points  $\frac{\log(c(p))}{\log p} + 2\pi i \log(p)k$  and as the function  $x \mapsto \frac{\log(c(x))}{\log(x)}$  is monotonically decreasing for  $x > 1$  we get the extension of  $L(s)$ .  $\square$

In the special case, when the rank of the group  $\mathbb{G}$  is one, we can say more. In this case we also get a result for small values of  $B$ , which we explain in the examples  $\mathbb{G} = \operatorname{PGL}_2$  and  $\operatorname{SL}_2$ .

**Theorem 5.3.** *For the group  $\mathbb{G} = \operatorname{PGL}_2$ , as  $T \rightarrow \infty$ ,*

$$N(T) \sim \frac{30}{\pi^2} \int_0^\infty e^{-2t} b^\infty(t) dt \frac{1}{\operatorname{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} e^{2T}.$$

*Equivalently we can write*

$$\pi(x) \sim \frac{30}{\pi^2} \int_0^\infty e^{-2t} b^\infty(t) dt \frac{1}{\operatorname{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} x^2.$$

**Corollary 5.4.** *For the group  $\mathbb{G} = \operatorname{SL}_2$  we get*

$$\pi(x) \sim \frac{15}{\pi^2} \int_0^\infty e^{-\frac{3}{2}t} b^\infty(t) dt \frac{1}{\operatorname{vol}(\mathbb{G}_{\mathbb{Q}} \backslash \mathbb{G}_{\mathbb{A}})} x^{3/2}.$$

*Proof.* Let  $\mathbb{G} = \operatorname{PGL}_2$ . Then we have

$$L(s) = \frac{\zeta(s)\zeta(s-1)}{\zeta(2s)}.$$

This function is holomorphic in  $\operatorname{Re}(s) > 2$  with a simple pole at  $s = 2$  of residue  $\zeta(2)/\zeta(4) = \frac{15}{\pi^2}$ . A straightforward application of the Wiener-Ikehara Theorem [Cha68] yields for any  $0 \leq B < 2$  that

$$\sum_{m \leq e^T} \frac{D(m)}{m^B} \sim \frac{15}{\pi^2} e^{(2-B)T},$$

as  $T \rightarrow \infty$ . On the other hand we have  $b^\infty(T) \sim e^{BT}$ . Suppose that  $0 < B < 2$ . Then we have two Radon measures  $\mu$  and  $\nu$  on  $[0, \infty)$  with  $\mu([0, T]) = \sum_{m \leq e^T} \frac{D(m)}{m^B} \sim \frac{15}{\pi^2} e^{(2-B)T}$  and  $\nu([0, T]) = b^\infty(T) \sim e^{BT}$ .

Lemma 2.5 implies that

$$\lim_{T \rightarrow \infty} \frac{b(T)}{e^{2T}} = \frac{15}{\pi^2} \int_0^\infty e^{-2t} d^\infty(t) dt,$$

which by Theorem 1.4 implies the claim of the theorem. In order to prove the corollary, one notes that  $\mathrm{SL}_2(\mathbb{Q}_p)$  does not act transitively on the vertices of the Bruhat-Tits tree, but has two orbits. Hence a calculation similar to the  $\mathrm{PGL}_2$ -case shows that

$$L(s) = \frac{\zeta(2(s-1))\zeta(2s-1)}{\zeta(4s-2)},$$

which has a simple pole at  $s = \frac{3}{2}$  of residue  $\frac{1}{2}$  which gives the claim.  $\square$

**Corollary 5.5.** *Let  $N(T)$  be the counting function of the group  $\mathbb{G} = \mathrm{PGL}_d$  with  $d \geq 3$ . If  $0 < B < d$ , we have for any  $\delta > 0$ , as  $T \rightarrow \infty$ ,*

$$e^{Ts_2} \ll N(T) \ll e^{T(s_2+1+\delta)},$$

where  $s_2 = \frac{\log(d2^{d-1}-2)}{\log 2}$ , see Lemma 5.2.

*Proof.* We have

$$L(s) = \prod_p \left( 1 + \sum_{k=1}^{\infty} D(p)c(p)^{k-1}p^{-ks} \right).$$

Write

$$\prod_{p \geq 3} \left( 1 + \sum_{k=1}^{\infty} D(p)c(p)^{k-1}p^{-ks} \right) = \sum_{m=1}^{\infty} \frac{D^{(2)}(m)}{m^s}.$$

Then

$$\begin{aligned}
\sum_{m \leq e^T} \frac{D(m)}{m^B} &= \sum_{2^k \leq e^T} \frac{D(2)c(2)^{k-1}}{2^{Bk}} \sum_{\substack{m' \leq e^T/2^k \\ 2 \nmid m'}} \frac{D^{(2)}(m')}{(m')^B} \\
&\geq \sum_{2^k \leq e^T} \frac{D(2)c(2)^{k-1}}{2^{Bk}} \\
&= \frac{D(2)}{c(2)} \sum_{k=0}^{\left\lfloor \frac{T}{\log 2} \right\rfloor} \left( \frac{c(2)}{2^B} \right)^k \\
&= \frac{D(2)}{c(2)} \frac{\left( \frac{c(2)}{2^B} \right)^{\left\lfloor \frac{T}{\log 2} \right\rfloor} - 1}{\frac{c(2)}{2^B} - 1} \\
&\geq \frac{D(2)}{c(2)} \frac{\left( \frac{c(2)}{2^B} \right)^{\frac{T}{\log 2} - 1} - 1}{\frac{c(2)}{2^B} - 1} \\
&= \frac{D(2)}{c(2)} \frac{\frac{2^B}{c(2)} e^{T(s_2 - B)} - 1}{\frac{c(2)}{2^B} - 1},
\end{aligned}$$

so

$$\sum_{m \leq e^T} \frac{D(m)}{m^B} \gg e^{T(s_2 - B)}.$$

On the other hand, by Lemma 5.2 we have that

$$\sum_{m \leq e^T} \frac{D(m)}{m^B} \ll e^{T(s_2 + 1 + \delta - B)}$$

for any  $\varepsilon > 0$ . With the methods of the proof of the last theorem we conclude

$$e^{Ts_2} \ll b(T) \ll e^{T(s_2 + 1 + \delta)},$$

as  $T \rightarrow \infty$ . Now the present corollary follows from Corollary 1.6 together with Corollary 2.6.  $\square$

## References

- [Art79] James Arthur, *Eisenstein series and the trace formula*, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 253–274.

- [BM90] V. V. Batyrev and Yu. I. Manin, *Sur le nombre des points rationnels de hauteur borné des variétés algébriques*, Math. Ann. **286** (1990), no. 1-3, 27–43, DOI 10.1007/BF01453564 (French).
- [Bor63] Armand Borel, *Some finiteness properties of adèle groups over number fields*, Inst. Hautes Études Sci. Publ. Math. **16** (1963), 5–30.
- [Cha68] K. Chandrasekharan, *Introduction to analytic number theory*, Die Grundlehren der mathematischen Wissenschaften, Band 148, Springer-Verlag New York Inc., New York, 1968.
- [Dei13] Anton Deitmar, *Automorphic forms*, Universitext, Springer, London, 2013. Translated from the 2010 German original.
- [DE14] Anton Deitmar and Siegfried Echterhoff, *Principles of harmonic analysis*, 2nd ed., Universitext, Springer, Cham, 2014.
- [FMT89] Jens Franke, Yuri I. Manin, and Yuri Tschinkel, *Rational points of bounded height on Fano varieties*, Invent. Math. **95** (1989), no. 2, 421–435, DOI 10.1007/BF01393904.
- [GMO08] Alex Gorodnik, François Maucourant, and Hee Oh, *Manin’s and Peyre’s conjectures on rational points and adelic mixing*, Ann. Sci. Éc. Norm. Supér. (4) **41** (2008), no. 3, 383–435 (English, with English and French summaries).
- [Kna86] Anthony W. Knap, *Representation theory of semisimple groups*, Princeton Mathematical Series, vol. 36, Princeton University Press, Princeton, NJ, 1986. An overview based on examples. MR855239 (87j:22022)
- [Kne66] Martin Kneser, *Strong approximation*, Algebraic Groups and Discontinuous Subgroups (Proc. Sympos. Pure Math., Boulder, Colo., 1965), Amer. Math. Soc., Providence, R.I., 1966, pp. 187–196.
- [Man91] Yu. I. Manin, *Three-dimensional hyperbolic geometry as  $\infty$ -adic Arakelov geometry*, Invent. Math. **104** (1991), no. 2, 223–243, DOI 10.1007/BF01245074.
- [Men15] Susanne Mennecke, *Höhenfunktionen auf halbeinfachen algebraischen Gruppen*, Thesis, University of Tübingen (2015), available at <https://publikationen.uni-tuebingen.de/xmlui/handle/10900/64356>.
- [Pie82] Richard S. Pierce, *Associative algebras*, Graduate Texts in Mathematics, vol. 88, Springer-Verlag, New York-Berlin, 1982. Studies in the History of Modern Science, 9.
- [Rap14] Andrei S. Rapinchuk, *Strong approximation for algebraic groups*, Thin groups and superstrong approximation, Math. Sci. Res. Inst. Publ., vol. 61, Cambridge Univ. Press, Cambridge, 2014, pp. 269–298.
- [STBT04] Joseph Shalika, Ramin Takloo-Bighash, and Yuri Tschinkel, *Rational points on compactifications of semi-simple groups of rank 1*, Arithmetic of higher-dimensional algebraic varieties (Palo Alto, CA, 2002), Progr. Math., vol. 226, Birkhäuser Boston, Boston, MA, 2004, pp. 205–233.
- [STBT07] ———, *Rational points on compactifications of semi-simple groups*, J. Amer. Math. Soc. **20** (2007), no. 4, 1135–1186, DOI 10.1090/S0894-0347-07-00572-3.
- [TBT13] Ramin Takloo-Bighash and Yuri Tschinkel, *Integral points of bounded height on compactifications of semi-simple groups*, Amer. J. Math. **135** (2013), no. 5, 1433–1448, DOI 10.1353/ajm.2013.0044.

- [Tit79] J. Tits, *Reductive groups over local fields*, Automorphic forms, representations and  $L$ -functions (Proc. Sympos. Pure Math., Oregon State Univ., Corvallis, Ore., 1977), Proc. Sympos. Pure Math., XXXIII, Amer. Math. Soc., Providence, R.I., 1979, pp. 29–69.
- [Wer02] Annette Werner, *Arakelov intersection indices of linear cycles and the geometry of buildings and symmetric spaces*, Duke Math. J. **111** (2002), no. 2, 319–355, DOI 10.1215/S0012-7094-02-11125-9.
- [Zim84] Robert J. Zimmer, *Ergodic theory and semisimple groups*, Monographs in Mathematics, vol. 81, Birkhäuser Verlag, Basel, 1984.

Mathematisches Institut  
Auf der Morgenstelle 10  
72076 Tübingen  
Germany  
`deitmar@uni-tuebingen.de`,  
`rupert-gordon.mccallum@uni-tuebingen.de`

January 5, 2016